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A continuum model of microheterogeneous media☆

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ABSTRACT

A correct model of media with a microstructure (according to Mindlin's definition), which is defined by the presence of free strains and generalizes the well-known Mindlin, Cosserat and Aero–Kuvshinskii models, is proposed. The correctness of the formulation of the model is determined by using a "kinematic" variational principle, based on a systematic formal description of the kinematics of media, formulation of the kinematic constraints for media of various complexity and the construction of the corresponding strain potential energy using a Lagrange multiplier procedure. A system of constitutive relations is established, and a consistent statement of the boundary-value problem is formulated. It is shown that the model of a medium investigated not only reflects scale effects that are similar to cohesive interactions, but also provides a basis for describing a broad spectrum of adhesive interactions. An interpretation of the physical characteristics responsible for non-classical effects is proposed in the context of an analysis of the physical aspects of the model, and a description of the spectrum of adhesion mechanical parameters is given. ©2009.

According to the previously proposed classification[1] of media with different defect fields, the model of a strained medium investigated here is a model of a medium with conserved dislocations. Applied versions of this model have led to explanations for numerous well-known non-classical phenomena in the mechanics of materials. For example, it has been shown that they enable researchers to successfully model the variation of the mechanical properties of nanocomposites with variation of the size of the reinforcing nanoparticles at a constant volumetric content,[2,3] as well as the dependence of the mechanical properties of thin films on their thickness.[4,5] Scale effects in the mechanics of materials associated with cohesive interactions have been modelled,[6-9] and a description of non-singular cracks with a flare angle equal to zero, which essentially gives a formal mathematical proof of Barenblatt's hypothesis regarding the existence of a cohesive field, has been proposed. Consideration of scale effects has enabled a consistent theory of an interfacial layer to be constructed that simulates local effects on boundaries between contacting phases.[2,3,7,8] A mathematical proof of the equivalent matrix hypothesis, the equivalent inclusions hypothesis, etc. has been obtained within this theory. Analytical estimates of the geometric and mechanical properties of the interfacial layer from classical and non-classical mechanical characteristics of the phases have been obtained.

A general version of a model of media with conserved dislocations (for which the dislocation flux through a closed surface of any volume is equal to zero), which generalizes the well-known Mindlin,[10,11] Cosserat,[12] Tupin[13] and Aero–Kuvshinskii[14] models, will be developed in this paper. A variational formulation of models based on the "kinematic" variational principle, previously formulated in Refs 15-17 and further developed in Refs 6-9, will be used. It will be shown that the spectrum of internal interactions is completely specified by the system of kinematic constraints realized in the medium. Therefore, the kinematic relations in a model of a medium, which enable us to formulate the kinematic constraints within the principle of possible displacements, will be investigated in the first stage of formulating the model. Note, for example, that in the classical theory of elasticity, the kinematics is completely specified by symmetrical Cauchy relations. In moment models of media with hindered rotation, the kinematics is assigned by a set of Cauchy relations and expressions that define derivatives of the rotation vector in terms of the displacement vector (see Refs 9, 11, 17, etc.). In the first stage, a list of arguments of the strain potential energy (for reversible processes) and the Lagrange functional will be established.[17] The general form of the constitutive equations corresponding to the general form of the potential energy will be presented, and these equations will be analysed, enabling us to introduce some simplifications associated with consideration of reported experimental data. A variational formulation of the boundary-value problem will be written as a result.

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1. The kinematic model

We will write the well-known relations for the components of the displacement vector R_i, which are obtained by formal integration of the asymmetrical Cauchy relations $d_{ij} = R_{i,j}$:

$$R_i = R_i^0 + \int_{M_0}^{M_x} d_{in} dx_n; \quad d_{in} = \gamma_{in} + \theta\delta_{in}/3 - \omega_k \Im_{ink} \tag{1.1}$$

Here γ_{in} denotes the components of the strain deviator tensor, θ denotes the volumetric strain, and ω_k denotes the components of the elastic rotation vector (pseudovector).

The quadratures of asymmetrical Cauchy relations (1.1) can be regarded as relations that specify the vector potential with the components R_i for a distortion tensor with the components d_{ij}.

We write the conditions for the Cauchy relations to be integrable relative to the displacement vector

$$d_{in,m}\Im_{nmj} = 0 \tag{1.2}$$

We will call homogeneous equations (1.2) homogeneous Papkovich equations because Papkovich was the first to appreciate the importance of these relations in the hierarchy of kinematic constraints in continuum mechanics. When equalities (1.2) hold, the displacement vector with the components R_i is the vector potential for the distortion tensor with the components d_{ij}^0:

$$d_{ij}^0 = R_{ij} \tag{1.3}$$

Note that the differential form $dR_i = d_{ij}^0 dx_j$ is a total differential.

Now consider the heterogeneous Papkovich equations

$$d_{in,m}\Im_{nmj} = \Xi_{ij} \tag{1.4}$$

The quantity Ξ_{ij} specifies the incompatibility of the displacements. Note that here the Ξ_{ij} are components of a second-rank pseudotensor, since the sign of each component changes when the right-hand set of three unit vectors is replaced by the left-hand set. In this case the displacement vector can be formally introduced as the difference between the displacements of two infinitely close points using the relation $dD_i = d_{ij}dx_j$. However, here the linear differential form dD_i will no longer be a complete differential, and the equation written for the displacements D_i will not be integrable. We will say that a vector with the components D_i specifies a defect displacement field. The continuous displacement “incompatibility” tensor with the components Ξ_{ij} is a dislocation density tensor[18] and obeys the differential conservation law

$$\Xi_{ij,j} = 0$$

The solution of heterogeneous equations (1.4) can be represented in the form of the sum of the solutions of the homogeneous equations (with the superscript 0) and the partial solution of heterogeneous equations (1.4) (with the superscript Ξ);

$$d_{ij} = d_{ij}^0 + d_{ij}^\Xi$$

The solution of homogeneous equations (1.2) can be written in terms of displacements in the form of the asymmetrical Cauchy relations $d_{ij}^0 = R_{i,j}$. We will represent the asymmetrical tensor with the components d_{ij}^0 in the form of an expansion into a deviator tensor with the components γ_{ij}^0, a spherical tensor with the components $\theta^0\delta_{ij}$ and an antisymmetrical tensor with the components $\omega_k^0\Im_{ijk}$. In turn, we will write the components of the antisymmetrical tensor in terms of the components ω_k^0 of the rotation pseudovector

$$d_{ij}^0 = \gamma_{ij}^0 + \theta^0\delta_{ij}/3 - \omega_k^0\Im_{ijk} \tag{1.5}$$

where

$$\gamma_{ij}^0 = R_{i,j}/2 + R_{j,i}/2 - R_{k,k}\delta_{ij}/3, \quad \omega_k^0 = -R_{i,j}\Im_{ijk}/2, \quad \theta^0 = R_{k,k}$$

There is no continuous vector potential for a particular solution of heterogeneous equations (1.4), i.e., it is impossible to represent the vector potential in the form (1.3). Only the following representation can be written for it:

$$d_{ij}^\Xi = \gamma_{ij}^\Xi + \theta^\Xi\delta_{ij}/3 - \omega_k^\Xi\Im_{ijk}$$

Clearly, along with d_{ij}^Ξ, the parameters γ_{ij}^Ξ, ω_k^Ξ and θ^Ξ can be regarded as independent “generalized variables.”

The general solution of heterogeneous equations (1.4) can be written in the symmetrised form

$$d_{ij} = d_{ij}^0 + d_{ij}^\Xi = \gamma_{ij} + \theta\delta_{ij}/3 - \omega_k\Im_{ijk}$$

Here

$$\gamma_{ij} = \gamma_{ij}^0 + \gamma_{ij}^\Xi = R_{i,j}/2 + R_{j,i}/2 - R_{k,k}\delta_{ij}/3 + \gamma_{ij}^\Xi$$

$$\omega_k = \omega_k^0 + \omega_k^\Xi = -R_{i,j}\Im_{ijk}/2 + \omega_k^\Xi, \quad \theta = \theta^0 + \theta^\Xi = R_{k,k}/3 + \theta^\Xi$$

Using Cosserat's terminology for the kinematics of media, we will call $\omega_k^0 = -R_{i,j}Э_{ijk}/2$ hindered rotation, and we will call ω_k^{Ξ} free rotation or spin. Similarly, we will call γ_{ij}^0 and θ^0 hindered strains, and we will call γ_{ij}^{Ξ} and θ^{Ξ} free strains. Accordingly, we will introduce the definitions of the components of the free distortion tensor d_{ij}^{Ξ} and the hindered distortion tensor d_{ij}^0.

The heterogeneous relations (1.4) and the Cauchy relations for hindered distortion (1.5) describe the kinematics of media with dislocation-type defects. We will call such media models Papkovich media or first-rank imperfect media.[1]

The kinematics of such media has the following structure.

1^0. The defect displacement field D_i is the result of the superposition of two fields, viz., the continuous field $D_i^1 = R_i$ (the displacements R_i) and the displacement discontinuity field D_i^2 (the dislocation field):

$$D_i = D_i^1 + D_i^2 = R_i + D_i^2$$

2^0. The displacement discontinuity field D_i^2 (the dislocation field) is expressed in integral form in terms of the free strain and spin fields according to formulae similar to the Cesàro formulae:

$$D_i^2 = \int_{M_0}^{M_x} d_{ij}^{\Xi} dy_j$$

However, unlike the Cesàro formulae, here the integrand does not satisfy the integrability conditions:

$$d_{ij,m}^{\Xi} Э_{nmj} = \Xi_{ij} \neq 0 \tag{1.6}$$

i.e., the curvilinear integral depends on the integration path, and, therefore, the vector field D_i^2 will not be continuous. Three types of dislocations ($(D_i^2)_\gamma$, $(D_i^2)_\theta$ and $(D_i^2)_\omega$) can be defined:

$$D_i^2 = \int_{M_0}^{M_x} d_{ij}^{\Xi} dy_j = \int_{M_0}^{M_x} \gamma_{ij}^{\Xi} dy_j + \frac{1}{3}\int_{M_0}^{M_x} \theta^{\Xi} dy_i + \left(-\int_{M_0}^{M_x} \omega_k^{\Xi} Э_{ijk} dy_j \right) = (D_i^2)_\gamma + (D_i^2)_\theta + (D_i^2)_\omega \tag{1.7}$$

3^0. The Cauchy relations generalized to imperfect media with dislocations, hold:

$$d_{ij} = D_{i,j} = D_{i,j}^1 + D_{i,j}^2 = R_{i,j} + d_{ij}^{\Xi}$$

4^0. The displacement "incompatibility" tensor with the components Ξ_{ij} is the dislocation tensor:[18]

$$\Xi_{ij} = d_{in,m}^{\Xi} Э_{nmj}$$

Three types of dislocation tensors with components related to γ_{ij}^{Ξ}, γ_k^{Ξ} and θ^{Ξ}, respectively, can be defined:

$$\Xi_{ij} = d_{in,m}^{\Xi} Э_{nmj} = \gamma_{in,m}^{\Xi} Э_{nmj} + \theta_m^{\Xi} Э_{imj}/3 - \omega_{k,m}^{\Xi} Э_{nki} Э_{nmj} = (\Xi_{ij})_\gamma + (\Xi_{ij})_\theta + (\Xi_{ij})_\omega \tag{1.8}$$

The quantities $(\Xi_{ij})_\gamma$, $(\Xi_{ij})_\theta$ and $(\Xi_{ij})_\omega$ in expansion (1.8) are sources of three types of dislocations, viz., γ, θ and ω dislocations, respectively.

5^0. A differential dislocation conservation law that follows from the definition of the dislocation tensor holds, since

$$\Xi_{ij,j} = 0$$

6^0. The integral analogue of the dislocation conservation law clearly has the form

$$\iiint \Xi_{ij,j} dV = \oint\!\!\oint \Xi_{ij} n_j dF = 0$$

Note that the flux of the tensor with the components Ξ_{ij} through the plane in which the planar contour chosen lies can be selected as a measure of the defect (dislocation) content:

$$\iint_F \Xi_{ij} n_j dF = n_j \iint_0 \Xi_{ij} dF$$

Here F is an arbitrary surface stretched onto the planar contour.

In other words, the flux of the tensor with components Ξ_{ij} through any surface stretched onto the planar contour is the same. No new dislocations are generated. For just this reason, here we can refer to the models as models of a medium with conserved dislocations.

If follows from the foregoing analysis (see also Ref. 1) that the concept of a defect in a continuous medium is complex and can be specified using a set of tensor objects. For dislocations such a set consists of the "incompatibility" pseudotensor with components Ξ_{ij}, the second-rank free distortion tensor with components d_{ij}^{Ξ} and the discontinuous displacement vector (first-rank tensor) with components D_i^2. The corresponding Burgers vector can also be included here. Its components can be obtained from equalities (1.7) by juxtaposing the initial point M_0 and the final point M_x of the planar integration path (n_n denotes the components of the constant vector of a normal to the plane of the integration path):

$$b_i = \int_{M_0}^{M_x} d_{ij}^{\Xi} dx_j \equiv \oint d_{ij}^{\Xi} dx_j = \oint d_{ij}^{\Xi} s_j ds = \oint d_{ij}^{\Xi} \upsilon_m n_n \ni_{jmn} ds$$

$$= \iint d_{ij,m}^{\Xi} n_n \ni_{jmn} dF = \iint \Xi_{in} n_n dF = n_n \iint \Xi_{in} dF$$

where s_j denotes the components of a unit vector that is tangential to the planar contour, n_n denotes the components of the vector of a unit normal to the plane of the path, and the vectors with the components s_j, ν_m and n_n form a set of three unit vectors that are attached to the current point on the contour.

A kinematic analysis of the model enables us to find the complete set of generalized "coordinates" and "velocities" that are needed to formulate the functional and the corresponding variational equation of the model of media. In the case under consideration of a Papkovich medium with a system of conserved dislocation defects, the continuous parameters R_i and d_{ij}^{Ξ} will be the generalized coordinates, and the corresponding tensor parameters with the components d_{ij}^0 and Ξ_{ij} should be regarded as the "velocities" of the kinematic state.

We also note that as a result of the kinematic analysis performed, a new natural classification of dislocations is essentially proposed. A classification of dislocations based on an invariant definition of slip dislocations

$$b_i \nu_i = \nu_i \oint d_{ij}^{\Xi} s_j ds, \quad b_i n_i = n_i \oint d_{ij}^{\Xi} s_j ds$$

and rupture dislocations

$$b_i s_i = s_i \oint d_{ij}^{\Xi} s_j ds$$

as the corresponding projections of the Burgers vector was previously proposed.[18] Note that such a classification does not reflect the energy independence of the kinds of dislocations identified.

Here we propose a different classification, which eliminates this drawback. We write the expression for the Burgers vector

$$b_i = \oint d_{ij}^{\Xi} s_j ds = \oint \gamma_{ij}^{\Xi} s_j ds + \frac{1}{3} \oint \theta^{\Xi} s_i ds - \oint \omega_k^{\Xi} \ni_{ijk} s_j ds = (b_i)_\gamma + (b_i)_\theta + (b_i)_\omega$$

Therefore, in accordance with the proposed classification, the quantities γ_{ij}^{Ξ} will be called γ dislocations, the quantity θ^{Ξ} will be called θ dislocations, and the quantities $\omega^{\Xi}{}_k$ will be called ω dislocations. It will be shown below that the potential energies of the free change in shape $\mu^{22}\gamma_{ij}^{\Xi}\gamma_{ij}^{\Xi}$, the free change in volume $(2\mu^{22}+3\lambda^{22})\theta^{\Xi}\theta^{\Xi}/6$, and twisting $\chi^{22}\omega_k^{\Xi}\omega_k^{\Xi}$ do not have cross terms. Therefore, the potential energies of the kinds of dislocations introduced are additive, and they can exist in isolation and independently of other dislocations.

2. Variational formulation of the model

The "kinematic" variational principle for constructing models of media was formulated in Refs. 15-17. According to this principle, the kinematic constraints in the medium are determined, and the possible work of the internal forces is postulated as the possible work of the reactive force factors on the kinematic constraints that are inherent in the medium. The possible work of the internal forces is represented in the form of the linear form of the variations of their arguments. This form can be integrated for conservative media. The potential energy is determined as a result. For linear media the potential energy is a quadratic form of its arguments.

For media with conserved dislocations, such kinematic constraints are the inhomogeneous Papkovich equations for free distortion and the homogeneous Papkovich equations for hindered distortion. The homogeneous Papkovich equations for hindered distortion can be integrated in general form. Their solutions are asymmetrical Cauchy relations. Thus, in accordance with the "kinematic" variational principle, the virtual work of the internal forces should be represented in the form

$$\overline{\delta U} = \iiint [\sigma_{ij}(d_{ij}^0 - R_{i,j}) + m_{ij}\delta(\Xi_{ij} - d_{in}^{\Xi}\ni_{nmj})] dV \tag{2.1}$$

Here $\overline{\delta U}$ is the virtual work, which, in the general case, is the linear form of the variations of its arguments (which is not necessarily integrable, as occurs for media with dissipation (see Ref. 19)), and the σ_{ij} and m_{ij} are the components of tensors of Lagrange multipliers, which have the physical meaning of the reactive force factors that ensure that the corresponding kinematic constraints hold.

We represent $\overline{\delta U}$ (2.1) as the linear form of the variations of its arguments. Using integration by parts in the terms containing derivatives, we obtain

$$\overline{\delta U} = \iiint [\sigma_{ij}\delta d_{ij}^0 + \sigma_{ij,j}\delta R_i + m_{ij}\delta\Xi_{ij} + m_{ij,m}\ni_{nmj}\delta d_{in}^{\Xi}] dV$$
$$+ \oiint [-\sigma_{ij} n_j \delta R_i - m_{ij} n_m \ni_{nmj} \delta d_{in}^{\Xi}] dF \tag{2.2}$$

We shall confine ourselves to media without energy dissipation (models of media with dissipation were previously considered[19]). Then there is a potential U (potential energy) that is such that the virtual work $\overline{\delta U}$ (2.2) will be the variation of this potential:

$$\overline{\delta U} = \delta U; \quad U = \iiint U_V dV + \oiint U_F dF, \quad U_V = U_V(d_{ij}^0; d_{ij}^{\Xi}; \Xi_{ij}; R_i), \quad U_F = U_F(d_{ij}^{\Xi}; R_i)$$

Henceforth we will exclude the displacement vector from the lists of arguments for the potential energy densities. Then the generalized model of a medium with scale effects under consideration will not contradict the existing experimental data in the special case of classical theory. This question will be discussed further below. As a result, we obtain

$$U = \iiint U_V dV + \oiint U_F dF; \quad U_V = U_V(d^0_{ij}; d^{\Xi}_{ij}; \Xi_{ij}), \quad U_F = U_F(d^{\Xi}_{ij}) \tag{2.3}$$

Taking into account the list of arguments in equality (2.3) and calculating the variation δU in the volume, we obviously obtain

$$\sigma_{ij} = \partial U_V/\partial d^0_{ij}, \quad m_{ij} = \partial U_V/\partial \Xi_{ij}, \quad p_{in} = m_{ij,m,}\Im_{nmj} = \partial U_V/\partial d^{\Xi}_{ij},$$
$$M_{ij} = \partial U_F/d^{\Xi}_{ij} = A_{ijnm} d^{\Xi}_{nm} \tag{2.4}$$

Formulae (2.4) should be interpreted as generalized Green's formulae for bulk and surface force factors. These relations enable us to write the variation of the Lagrangian

$$\delta L = \iiint [(\sigma_{ij,j} + X_i)\delta R^0_i - (m^{\Xi}_{in,m}\Im_{nmj} + p_{ij})\delta d^{\Xi}_{ij}] dV$$
$$+ \oiint [(Y_i - \sigma_{ij} n_j)\delta R^0_i - (M_{in} + m_{ij} n_m \Im_{nmj})\delta d^{\Xi}_{in}] dF \tag{2.5}$$

and to find the corresponding Euler equations.

3. Constitutive relations. Physical interpretation of the generalized elastic constants

Consider again the potential energy densities in the bulk and on the surface. We will confine ourselves to considering physically linear media. Then U_V is defined as the quadratic form of its arguments:

$$2U_V = 2U_V(d^0_{ij}; d^{\Xi}_{ij}; \Xi_{ij}; R_i) = C^{11}_{ijnm} d^0_{ij} d^0_{nm} + 2C^{12}_{ijnm} d^0_{ij} d^{\Xi}_{nm} + C^{22}_{ijnm} d^{\Xi}_{ij} d^{\Xi}_{nm} + C^{33}_{ijnm} \Xi_{ij} \Xi_{nm} \tag{3.1}$$

The following fully justified simplifications were introduced when deriving this equality.

1^0. In the expression for the potential energy density (3.1) the coefficient in front of the term $C_{ij}R_iR_j$ is assumed to be equal to zero. Otherwise, the operator of the balance equations would have the form of the Helmholtz equations, ruling out the existence of homogeneous stress-strain states.
2^0. The coefficients in front of all the remaining bilinear components, that include the displacement vector as a cofactor, were also assumed to be equal to zero. Otherwise, in the absence of the term containing the quadratic form for the displacement vector $C_{ij}R_iR_j$, the bulk potential energy density would not be positive-definite.

The structure of the elastic modulus tensors C^{pq}_{ijnm} in equality (3.1) is specified by their expansion in fourth-rank isotropic tensors constructed as the product of a pair of Kronecker tensors with all possible permutations of the subscripts:

$$C^{pq}_{ijnm} = C^{pq}_1 \delta_{ij}\delta_{nm} + C^{pq}_2 \delta_{in}\delta_{jm} + C^{pq}_3 \delta_{im}\delta_{jn} \tag{3.2}$$

In order to give a physical interpretation of the elastic modulus tensors in equalities (3.1) and (3.2), we will analyse the corresponding fractions of the potential energies. Consider the bulk potential energy density $C^{11}_{ijnm} d^0_{ij} d^0_{nm}$, which is associated with invariants of the hindered distortion tensor with the components $d^0_{ij} = R_{i,j}$. We will represent the second-rank kinematic displacement tensor d^0_{ij} in the form of a tensor expansion into deviator, spherical and antisymmetrical parts:

$$d^0_{ij} = \gamma^0_{ij} + \theta^0\delta_{ij}/3 + \omega^0_{ij}; \quad \omega^0_{ij} = -\omega^0_k \Im_{ijk} \tag{3.3}$$

Then we obtain the equality

$$C^{11}_{ijnm} d^0_{ij} d^0_{nm} = [C^{11}_1 \delta_{ij}\delta_{nm} + C^{11}_2 \delta_{in}\delta_{jm} + C^{11}_3 \delta_{im}\delta_{jn}] d^0_{ij} d^0_{nm}$$
$$= (C^{11}_2 + C^{11}_3)\gamma^0_{nm}\gamma^0_{nm} + (3C^{11}_1 + C^{11}_2 + C^{11}_3)\theta^0\theta^0/3 + (C^{11}_2 - C^{11}_3)\omega^0_{nm}\omega^0_{nm} \tag{3.4}$$

The first term on the right-hand side of the last equality specifies the potential energy of the change in shape

$$(C^{11}_2 + C^{11}_3)\,(R_{i,j}/2 + R_{j,i}/2 - R_{k,k}\delta_{ij}/3)(R_{i,j}/2 + R_{j,i}/2 - R_{k,k}\delta_{ij}/3)$$

Therefore, we will call the corresponding multiplier the shear modulus for hindered distortion. Similar transformations also hold for the remaining terms in the bulk potential energy density $2C^{12}_{ijnm} d^0_{ij} d^{\Xi}_{nm}$, $C^{22}_{ijnm} d^{\Xi}_{ij} d^{\Xi}_{nm}$, $C^{33}_{ijnm} \Xi_{ij} \Xi_{nm}$. As a result, we can define analogues of the shear modulus μ^{pq} for the deviators of all the corresponding kinematic factors

$$(C^{pq}_2 + C^{pq}_3) = 2\mu^{pq} \tag{3.5}$$

The second term on the right-hand side of equality (3.4) specifies the potential energy of the change in volume

$$(3C^{11}_1 + C^{11}_2 + C^{11}_3) R_{k,k} R_{q,q} \delta_{ij}/3$$

Therefore, we will call the corresponding multiplier the bulk compression modulus for hindered distortion. Similarly, we introduce the bulk compression moduli $2\mu^{pq}+3\lambda^{pq}$ for the spherical tensors of all the remaining kinematic factors

$$(3C_1^{pq}+C_2^{pq}+C_3^{pq})=K^{pq}=2\mu^{pq}+3\lambda^{pq} \tag{3.6}$$

The third term on the right-hand side of equality (3.4) corresponds to the potential energy of twisting

$$(C_2^{11}-C_3^{11})(R_{i,j}/2-R_{j,i}/2)(R_{i,j}/2-R_{j,i}/2)$$

It specifies the asymmetry in the stress tensor and does not have classical analogues. We will call the corresponding multiplier the third Lamé coefficient for hindered distortion (d_{ij}^0). Accordingly, in the general case the analogues of the third Lamé coefficient χ^{pq} are defined by the formulae

$$(C_2^{pq}-C_3^{pq})=2\chi^{pq} \tag{3.7}$$

Solving system of equations (3.5)–(3.7) for the coefficients $C_j^{pq}(j=1,2,3)$, taking equality (2.2) into account, we obtain

$$C_{ijnm}^{pq}=\lambda^{pq}\delta_{ij}\delta_{nm}+(\mu^{pq}+\chi^{pq})\delta_{in}\delta_{jm}+(\mu^{pq}-\chi^{pq})\delta_{im}\delta_{jn},\quad C_{ijnm}^{pq}=C_{ijnm}^{qp} \tag{3.8}$$

Finally, we write the following expression for the potential energy density

$$\begin{aligned}U_V &= \mu^{11}(\gamma_{nm}^0\gamma_{nm}^0)+2\mu^{12}(\gamma_{nm}^0\gamma_{nm}^\Xi)+\mu^{22}(\gamma_{nm}^\Xi\gamma_{nm}^\Xi)+\\ &+[(2\mu^{11}+3\lambda^{11})\theta^0\theta^0+2(2\mu^{12}+3\lambda^{12})\theta^0\theta^\Xi+(2\mu^{11}+3\lambda^{11})\theta^\Xi\theta^\Xi]/6\\ &+\chi^{11}\omega_{nm}^0\omega_{nm}^0+2\chi^{12}\omega_{nm}^0\omega_{nm}^\Xi+\chi^{22}\omega_{nm}^\Xi\omega_{nm}^\Xi+C_{ijnm}^{33}\Xi_{ij}\Xi_{nm}/2\end{aligned}$$

Note that the part of the strain energy density that is associated with the dislocation tensor (with the components $C_{ijnm}^{33}\Xi_{ij}\Xi_{nm}$) specifies the rapidly varying local part of the potential energy of the dislocations. The remaining part of the strain energy density varies slowly and is specified as the sum of the potential energies of three types of dislocations: γ, θ and ω dislocations. The slowly varying part of the strain energy (with the exception of $C_{ijnm}^{33}\Xi_{ij}\Xi_{nm}$) does not contain cross terms from the types of dislocations just indicated and is an additive form relative to the components of the free distortion. For approximate estimates of the defect content in media when integral characteristics are used, the local, rapidly varying part of the energy can probably be neglected.

Note that the question of the material objectivity of the asymmetrical model of a medium whose potential energy density contains the terms $\chi^{11}\omega_{nm}^0\omega_{nm}^0$ (see also equality (3.4)) has been discussed repeatedly and was described in detail in Refs 7 and 17.

In the general case the bulk potential energy density does not contain cross terms corresponding to the free change in shape (γ_{nm}^Ξ), the change in volume (θ^Ξ), and twisting (ω_k^Ξ) for small values of the constants C_{ijnm}^{33}, which correspond to scale effects (their dimensions differ from the dimensions of Young's modulus by the square of the length). This fact served as proof of the correctness of the new classification of the different types of dislocations.

We will write the generalized Hooke's law equations (2.4) for the bulk force factors in the form

$$\sigma_{ij}=C_{ijnm}^{11}R_{n,m}+C_{ijnm}^{12}d_{nm}^\Xi,\quad p_{ij}=C_{ijnm}^{21}R_{n,m}+C_{ijnm}^{22}d_{nm}^\Xi,\quad m_{ij}=C_{ijnm}^{33}\Xi_{nm} \tag{3.9}$$

Note that the generalized momenta σ_{ij}, p_{ij} and m_{ij} in equality (3.9) depend not only on the generalized velocities $R_{n,m}$ and Ξ_{ij}, but also on the generalized coordinates d_{ij}^Ξ. For this reason, different interpretations of the "non-classical" components in the generalized Hooke's law (3.9) are possible.

On the one hand, the stress tensor can be redefined by eliminating the terms containing the free distortions d_{ij}^Ξ on the right-hand sides of equalities (3.9). The combination $C_{pqij}^{22}\sigma_{ij}-C_{pqij}^{12}p_{ij}$ can then serve as the components of the generalized stress tensor. The other linearly independent combination $C_{pqij}^{21}\sigma_{ij}-C_{pqij}^{11}p_{ij}$ will then have the physical meaning of the reaction of the generalized Winkler base to the generalized displacements d_{ij}^Ξ.

Under another interpretation of the constitutive relations it should be acknowledged that along with the stress tensor with the components σ_{ij} there are additional force factors, viz., the "dislocation" stresses p_{ij}, in such media. This alternative is more traditional[11] and preferable. We offer the following arguments. We assume that $C_{ijnm}^{12}=0$. As will be shown below, in this case the general boundary-value problem breaks down into the boundary-value problem for the displacements R_i and the boundary-value problem for the free distortion d_{ij}^Ξ. Then the boundary-value problem for the displacements under the additional assumption that $\chi^{11}=0$ (the theory of elasticity with a symmetrical stress tensor) is identical to the classical theory of elasticity. The force factor σ_{ij} takes on the meaning of classical stresses. Accordingly, the force factor p_{ij} takes on the meaning of the Winkler reaction in the balance equations of the moment stresses. When $C_{ijnm}^{12}\neq 0$, mutual perturbation of the classical displacement field and the pure dislocation states occurs. The cross terms in the Hooke's law equations for σ_{ij} and p_{ij} reflect these perturbations. The same arguments clearly lead to an algorithm for solving the general boundary-value problem by the method of successive approximations.

The situation with the surface potential energy density is more complex. For a smooth surface there is always a naturally identifiable direction, namely, a normal to the surface. The Hooke's law equations for the internal force factors on the surface should have a transversal isotropic character, and, as a result, the kinematic factors associated with a normal to the surface and with the tangential plane will appear in these Hooke's law equations with different values.

We will examine the surface strain energy density in greater detail. Consider the expression for the surface part of the possible work. The first term in it corresponds completely to the classical representation. It appears as a result of integration by parts of the expression $\int\int\int[\sigma_{ij}\delta(R_{i,j})]dV$ in equality (2.1). The second term in the expression for the surface part of the possible work (2.2), (2.3) is non-classical.

Its appearance is due to the kinematic variational method for constructing the model and is associated with the surface adhesion energy U_F. We will examine this term in greater detail. We have

$$-\oiint m_{ij} n_m \ni_{nmj} \delta d_{in}^{\Xi} dF = -\oiint m_{ij} n_q \ni_{pqj} \delta[d_{nm}^{\Xi}\eta_{in} + d_{nm}^{\Xi} n_i n_n]\eta_{np} dF$$
$$-\oiint m_{ij} n_q \ni_{pqj} \delta[d_{nm}^{\Xi}\eta_{in} + d_{nm}^{\Xi} n_i n_n] n_m n_p dF \quad (3.10)$$

where $\eta_{in} = (\delta_{in} - n_i n_n)$ denotes the components of the "planar" Kronecker tensor.

We note that $n_p n_q \ni_{pqj} = 0$, since it is the convolution of the symmetrical tensor whose components are $n_p n_q$ with the antisymmetrical tensor whose components are $\ni_{pqj}$. Therefore, the work of the moment stresses (3.10) on the surface of the body is performed not on all nine components of the free distortion tensor d_{in}^{Ξ}, but only on six of them, $d_{im}^{\Xi}\eta_{pm}$:

$$-\oiint m_{ij} n_m \ni_{nmj} \delta d_{in}^{\Xi} dF = -\oiint m_{ij} n_q \ni_{pqj} \delta d_{im}^{\Xi} \eta_{pm} dF \quad (3.11)$$

In the general case the strain energy density (the adhesion potential energy) on the body surface has the form

$$U_F = [A_{ijnm} d_{nm}^{\Xi} d_{ij}^{\Xi}]/2 \quad (3.12)$$

The constitutive relations on the body surface are specified by equalities (2.4).

Note that the potential energy density does not depend on the displacement vector. Otherwise, the variational formulation would lead to systematic errors in the static boundary conditions in the classical solution, which would contradict existing experimental data.

It is noteworthy that relations (3.11) enable us to refine the list of arguments of the surface potential energy density (3.12). This refined list of arguments is now specified by six "planar" components of the free distortion tensor $d_{im}^{\Xi}\eta_{pm}$: $U_F = U_F(d_{ik}^{\Xi}\eta_{kj})$. As a result, the complete correct expression for the variation of the Lagrangian is distinguished from (2.5) by the replacement of δd_{in}^{Ξ} by $\delta d_{ik}^{\Xi}\eta_{kn}$.

Hence it follows that there are nine boundary conditions at each non-special point for the model of media under investigation. An analysis of the resolvent equations and the boundary-value problem as a whole enables us to show that the total order of the resolvent equations in the components of the displacement vector and the potentials for the components of the free distortion is equal to 18. Therefore, the mathematical formulation of the model investigated is consistent, since there are nine boundary conditions for the eighteenth-order boundary-value problem.

The structure of the adhesive modulus tensor A_{ijnm} is specified by its expansion in the fourth-rank tensors constructed as all possible products of pairs of "planar" Kronecker tensors and the tensors formed by products of the components of the vectors of the unit normal of the form $n_i n_j$ with all possible permutations of the subscripts. In addition, we will take into account that the adhesion potential energy should not depend on the following components of the free distortion tensor: $d_{ij}^{\Xi} n_j$. Only in this case do the classical, natural boundary conditions on the surface of the body for the stresses remain unchanged. This does not introduce contradictions in the special case of the transition to the classical model of a medium and is consistent with numerous experimental data. It can be shown that in such a case the overall structure of the components of the adhesion modulus tensor has the form

$$A_{ijnm} = A_1\eta_{ij}\eta_{nm} + A_3 n_i n_n \eta_{jm} + (A_2 + A_4)\eta_{in}\eta_{jm} + (A_2 - A_4)\eta_{im}\eta_{jn} \quad (3.13)$$

Here the A_i are constants.

We will examine the strain energy density on the surface and give a physical interpretation of the adhesion components of the strain energy taking into account relations (3.11) and (3.13). The surface potential energy will be a quadratic function only of the components of the free distortion of the form $d_{ik}^{\Xi}\eta_{jk} = d_{ik}^{\Xi}(\delta_{jk} - n_j n_k)$. This is important for constructing a consistent theory and will be discussed below.

Were represent the free distortion in the form of a tensor expansion in the "planar" deviator with the components

$${}^2\gamma_{ij} = d_{nm}^{\Xi}\eta_{in}\eta_{jm}/2 + d_{nm}^{\Xi}\eta_{jn}\eta_{im}/2 - d_{nm}^{\Xi}\eta_{ij}\eta_{nm}/2$$

the "planar" spherical tensor with the components

$${}^2\theta = d_{nm}^{\Xi}\eta_{nm}$$

the "planar" antisymmetrical tensor with the components

$${}^2\omega_{ij} = d_{nm}^{\Xi}\eta_{in}\eta_{jm}/2 - d_{nm}^{\Xi}\eta_{jn}\eta_{im}/2$$

and the "planar" flexural rotation angle vector of the surface with the components

$${}^2\alpha_i = d_{nm}^{\Xi} n_n \eta_{mi}$$

The left superscript "2" stresses the fact that the corresponding components of the free distortion tensor are calculated on the surface of the body. As a result, we obtain

$$d_{ik}^{\Xi}\eta_{jk} = {}^2\gamma_{ij} + {}^2\theta\eta_{ij}/2 + {}^2\omega_{ij} + {}^2\alpha_j n_i \quad (3.14)$$

Taking into account equality (3.13), we can see that the surface free distortion tensor represented in the form of expansion (3.14), converts U_F into the canonical form

$$2U_F = A_{ijnm} d_{ij}^{\Xi} d_{nm}^{\Xi} = (A_1 + A_2)({}^2\theta\,{}^2\theta) + 2A_2({}^2\gamma_{ij}\,{}^2\gamma_{ij}) + 2A_4({}^2\omega_{ij}\,{}^2\omega_{ij}) + A_3({}^2\alpha_k\,{}^2\alpha_k) \quad (3.15)$$

Comparing the first three terms on the right-hand side of equality (3.15) with the corresponding terms in the expression for the bulk potential energy in the planar formulation, we can draw analogies between the Lamé coefficients and the adhesion moduli.

We introduce the following natural definitions of the adhesion moduli.

A_2 is the adhesion analogue of the shear modulus: $A_2 = \mu^F$;
A_1 is the adhesion analogue of the second Lamé coefficient: $A_1 = \lambda^F$;
A_4 is the adhesion analogue of the third Lamé coefficient: $A_4 = \chi^F$;

$A_3 = \delta^F$ is the adhesion analogue of the Winkler stiffness of the "internal sublayer" of the surface, which gives rise to a reactive torque proportional to the free rotations of elements of the midline of the surface (presurface layer) in two orthogonal directions.

Accordingly, the first term in the expression for the potential energy (3.15) is the energy of the change in the "planar volume" of the surface. We will call this energy the surface tension energy. The theory of surface tension is fairly well known outside of continuum mechanics as an autonomous empirical theory. Thus, it can be asserted that the surface tension is a specific effect of the theory developed here.

The second and third terms in expression (3.15) specify the energy of the change in shape and the energy of twisting in a plane tangential to the surface, respectively. From the point of view of identifying physical constants of adhesion, we can jointly take into account the energies of the change in shape and twisting within test problems that simulate the static friction of two half-spaces with an ideally smooth contact surface. The problem of extruding a nanofiber from a "contracted" die can be proposed as the first test problem, for which only the energy of the change in shape is realized. By virtue of the axial symmetry of this problem, the surface of the nanofiber does not undergo torsional strains. The solution of such a problem establishes the relation between the static friction coefficient and the adhesion modulus. The antiplanar contact problem can be considered as the second problem. For this problem, a change in shape of the contact surface apparently occurs along with the twisting. The solution of such problems provides the possibility, in principle, of setting up corresponding experiments and to determine the adhesion moduli $A_2 = \mu^F$ and $A_4 = \chi^F$.

The fourth term specifies the bending energy of the surface, since it is the strain energy of the "internal Winkler springs." The problem of the behaviour of a medium that is in the gap between two half-spaces with ideally smooth surfaces and under pressure from the surfaces can be a test problem here. The use of a non-classical adhesion model here enables us to simulate two phenomena, viz., the existence of a meniscus on the surface of the medium and a phenomenon that is associated with the interpretation of the difference between the mean elongation of the medium in the gap found using the non-classical model and the analogous result obtained using the classical model.

4. The fundamental role of the cross tensor of the moduli

Consider again the constitutive relations (2.4) and (3.9). After setting $C^{12}_{ijnm} = 0$ in them, we obtain

$$\sigma_{ij} = C^{11}_{ijnm} R_{n,m}, \quad p_{ij} = C^{22}_{ijnm} d^{\Xi}_{nm}, \quad m_{ij} = C^{33}_{ijpq} \Xi_{pq} \tag{4.1}$$

It is seen that in this case the overall boundary-value problem breaks down into two independent boundary-value problems. Taking into account relations (3.9) and (4.1), we obtain the separate problem of determining the displacement vector

$$\iiint C^{11}_{ijnm}(R_{n,jm} + P^V_i)\delta R_i dV + \oint\!\!\oint (P^F_i - C^{11}_{ijnm} n_j R_{n,m})\delta R_i dF = 0 \tag{4.2}$$

The problem of determining the free distortion tensor can also be formulated separately:

$$\iiint [C^{22}_{ijnm} d^{\Xi}_{nm} - (C^{33}_{inpq} \ni_{nmj} \ni_{srq}) d^{\Xi}_{ps,mr}]\delta d^{\Xi}_{ij} dV$$
$$- \oint\!\!\oint [A_{ijnm} d^{\Xi}_{nm} - (C^{33}_{inpq} \ni_{nmj} \ni_{srq}) n_m d^{\Xi}_{ps,r}]\delta d^{\Xi}_{ij} dF = 0$$

Therefore, the boundary-value problem relative to the components of the free distortion tensor is homogeneous in this case. This corresponds to the absence of dislocations. As a result, when $C^{12}_{ijnm} = 0$, the model reduces to the model of a dislocation-free medium. The inhomogeneous subsystem of force balance equations (for $\chi^{11} = 0$) and boundary-value problem (4.2) are identical as a whole to the boundary-value problem of the classical theory of elasticity. The arguments presented enable us to give the following natural interpretations to the elastic moduli μ^{11}, $2\mu^{11} + \lambda^{11}$ and χ^{11}:

μ^{11} is the shear modulus of a medium that is free of γ dislocations, $\mu^{11} = G$;
$2\mu^{11} + 3\lambda^{11} = K$ is the bulk compression modulus of a medium that is free of θ dislocations, Young's modulus is then defined by the formula $2\mu^{11} + \lambda^{11} = E$;
χ^{11} can be interpreted as the "torsional modulus" of a medium that is free of ω dislocations: $\chi^{11} = \chi$.

We will next examine the equations obtained from variational equation (2.5). In order to find the generalized balance equations in displacements, we should set the multipliers in front of the variations of the displacements and in front of the variations of the components of the free distortion tensor equal to zero. We will call the first group of equations force balance equations. We will call the second group (the second-rank tensor equation) torque balance equations. The system of generalized balance equations can be written in the kinematic variables R_i and d^{Ξ}_{ij} using the generalized Hooke's law equations (3.9). From this system of equations we can isolate a subsystem of equations that generalize the Lamé balance equations of the classical theory of elasticity and are written in terms of a displacement vector with the components R_i:

$$E_1 R_{k,ik} + P_i^V - l_E^2 E_2 \Delta R_{k,ik} - l_E^2 P_{k,ik}^V$$
$$+G_1(\Delta R_i - R_{k,ik}) - l_G^2 G_2(\Delta R_i - R_{k,ik}) - l_G^2(\Delta P_i^V - P_{k,ik}^V) = 0$$

Here we use the following notation

$$E_1 = \frac{4}{3}\frac{\mu^{11}\mu^{22} - \mu^{12}\mu^{12}}{\mu^{22}} + \frac{1}{3}E_2, \quad E_2 = \frac{(2\mu^{11} + \lambda^{11})(2\mu^{22} + \lambda^{22}) - (2\mu^{12} + \lambda^{12})^2}{2\mu^{22} + \lambda^{22}}$$
$$G_1 = \frac{\mu^{11}\mu^{22} - \mu^{12}\mu^{12}}{\mu^{22}} + \frac{\chi^{11}\chi^{22} - \chi^{12}\chi^{12}}{\chi^{22}}, \quad G_2 = \frac{(\mu^{11} + \chi^{11})(\mu^{22} + \chi^{22}) - (\mu^{12} + \chi^{12})^2}{\mu^{22} + \chi^{22}}$$
$$l_E^2 = \frac{\chi^{33}(2\mu^{22} + \lambda^{22})}{\mu^{22}(2\mu^{22} + 3\lambda^{22})}, \quad l_G^2 = \frac{(\mu^{33} + \chi^{33})(\mu^{22} + \chi^{22})}{4\mu^{22}\chi^{22}} \tag{4.3}$$

The quantities E_1 and G_1 can clearly be interpreted as the elastic moduli of a medium with dislocations; the parameters l_G^2 and l_E^2 are scale characteristics of the medium, and they have the dimension of the square of the length.

Relations (4.3) lead directly to the inequalities

$$2\mu^{11} + \lambda^{11} - E_1 = \frac{4\mu^{12}\mu^{12}}{3\mu^{22}} + \frac{(2\mu^{12} + 3\lambda^{12})^2}{3(2\mu^{22} + 3\lambda^{22})} > 0$$
$$\mu^{11} + \chi^{11} - G_1 = \frac{\mu^{12}\mu^{12}}{\mu^{22}} + \frac{\chi^{12}\chi^{12}}{\chi^{22}} > 0 \tag{4.4}$$

Crucial relations (4.3) and (4.4) reveal the fundamental role of the cross tensor of the moduli. In addition, equalities (4.3) establish the exact relation between the elastic moduli of a medium that is dislocation-free, viz., $(2\mu^{11} + \lambda^{11})$ and $(\mu^{11} + \chi^{11})$, and the elastic moduli of a medium with dislocations, viz., E_1 and G_1.

It follows from inequalities (4.4) that the moduli of a medium with dislocations are always smaller than the elastic moduli of a medium that is dislocations-free. Equality occurs only when $\mu^{12} = 0$ and $\lambda^{12} = 0$ ($C_{ijnm}^{12} = 0$), and a medium with conserved dislocations degenerates into a classical medium that is dislocations-free (when $\chi^{11} = 0$).

It is noteworthy that the dimensions of the moduli μ^{33}, λ^{33}, χ^{33} differ from the dimensions of μ^{11}, λ^{11}, χ^{11}, μ^{12}, λ^{12}, χ^{12} and μ^{22}, λ^{22}, χ^{22} by the dimension of the square of the length. Thus, consideration of the contribution of the invariants of the dislocation pseudotensor $C_{ijnm}^{33}\Xi_{nm}\Xi_{ij}$ in the expression for the potential energy inevitably leads to scale effects in the bulk. We also note that the dimensions of the adhesion moduli differ from the bulk moduli by the dimension of length. Thus, consideration of the adhesion component in the expression for the potential energy leads to modelling of the scale effects on the surface.

As a whole, the generalized model of continuum mechanics presented is a theoretical model, in which surface tension, the static friction between two bodies with an ideally smooth contact surface, a meniscus, wettability and capillarity are modelled as specific effects within a single continuum description. All these specific effects are united by one characteristic feature, i.e., they are scale effects in continuum.

5. Summary and conclusions

A complete and correct model of media with conserved dislocations has been given on the basis of the kinematic variational approach.[2-4,6-9,17] Special attention has been focused on the analysis of the kinematic relations, because in a variational description the kinematics of the medium completely determines the system of internal interactions in the bulk and on the surface of the body under consideration. A new classification of dislocations, which enables us to identify three types of dislocations, viz., γ, θ and ω dislocations, has been proposed on the basis of the kinematic analysis conducted.

This classification has enabled us to propose a new kinematic interpretation for dislocations, which reflects the relation of dislocations to a change in shape γ, to a change in volume θ (porosity) and to twisting ω (rotations or spins). The proposed classification actually enables us to predict special cases of dislocations in which only one or two types of dislocations predominate in the medium.

For example, dislocations generated only by free rotations ω_k^{Ξ} can be predominant in a medium with distributed defects. Then we obtain the "classical" version of the Cosserat model of a medium, in which $\gamma_{ij}^{\Xi} = 0$, $\theta^{\Xi} = 0$, and the free distortion tensor is specified by the relation $d_{ij}^{\Xi} = -\omega_k^{\Xi}\ni_{ijk}$, as a special case of the general model.

A porous medium can also be regarded as a special case of the general model. Dislocations generated only by a free change in volume θ^{Ξ} predominate in a porous medium. Then for a porous medium with the four degrees of freedom R_i, θ^{Ξ} we have

$$\omega_k^{\Xi} = 0, \quad \gamma_{ij}^{\Xi} = 0, \quad d_{ij}^{\Xi} = \theta^{\Xi}\delta_{ij}/3$$

Finally, a medium with the eight degrees of freedom R_i, γ_{ij}^{Ξ} is also a special model with one predominant type of dislocations. Here dislocations generated only by a free change in shape γ_{ij}^{Ξ} predominate. In this case we clearly have $d_{ij}^{\Xi} = \gamma_{ij}^{\Xi}$.

The existence of media with two types of dislocations can be predicted in a similar manner. A model of such a medium, in which the dislocations generated by a free change in shape γ_{ij}^{Ξ} were neglected, was previously considered.[2-4,7] A continuum version of the theory of interfacial interactions was constructed on its basis. With this theory, well-known scale effects in the mechanics of finely dispersed composites that are specified by local cohesive and adhesive interactions could be modelled and explained. The classification proposed in this paper enables us to consider one more special case of media in which either θ^{Ξ} (non-porous media) or ω_k^{Ξ} (zero-spin media) can be neglected. As far as we know, such media have not yet been investigated.

The proposed classification has also been substantiated from the physical point of view, since it reflects the physical meaning of the different types of dislocations. For example, it was shown that these types of dislocations, viz., γ, θ and ω dislocations, make up corresponding, mutually independent component fractions of the main, slowly varying part of the strain energy density. These fractions of the potential energy do not have cross terms. Thus, there is additivity in the expansion of the slowly varying part of the strain energy density relative to the three different types of dislocations. The presence of a non-classical, non-local component part of the potential energy, which is associated with dislocation defects, is highly unexpected for a gradient model, such as a model of media with a system of distributed dislocations. Finally, it should be noted that the kinematics of the model examined and the proposed classification are entirely consistent with the general tenets of the geometrical theory of defects previously derived in Ref. 1.

The use of a systematic variational approach in this paper and the detailed analysis of the boundary conditions has enabled us to formulate a consistent and harmonious boundary-value problem for media with conserved dislocations with nine boundary conditions at each non-special point on the surface. It should be recalled that consistency of the mathematical formulation is always achieved as a result of a systematic variational formulation of the problem. The order of the problem is, in fact, specified by the number of independent boundary conditions. We noted that the theory proposed in this paper is fairly complete as a theory of media with a continuous field of conserved dislocations, although it formally corresponds to a more specific model compared with the theory of media with Mindlin microstructures.[11] In Mindlin's theory[11] the structure of the surface potential energy density corresponding to adhesive interactions was not investigated at all, and no analysis of the function of these surface interactions was given.

Finally, it should be noted that the spectrum of scale effects in the volume and on the surface is taken into account within the proposed model. In fact, consideration of the invariants of the dislocation pseudotensor $C^{33}_{ijnm}\Xi_{nm}\Xi_{ij}$ in the expression for the potential energy inevitably leads to scale effects in the volume. On the other hand, consideration of the adhesion energy in the expression for the potential energy leads to modelling of scale effects on the surface, since the dimension of the tensor with the components A_{ijnm} differs from the dimension of Young's modulus. The generalized model of continuum mechanics presented can obviously be regarded as the first correct theoretical model in which different special scale effects (cohesive interactions, surface tension etc.) in the volume and on the surface are modelled within a single continuum description.

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